

Embeddings of right-angled Artin groups into higher dimensional Thompson groups

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Abstract

In this paper, we construct embeddings of right-angled Artin groups into higher dimensional Thompson groups. In particular, we embed every right-angled Artin groups into n -dimensional Thompson groups, where n is the number of complementary edges in the defining graph. It follows that $\mathbb{Z}^n * \mathbb{Z}$ embeds into nV for every $n \geq 1$.

1 Introduction

The Thompson group V is an infinite simple finitely presented group, which is described as a subgroup of the homeomorphism group of the Cantor set C . Brin [1] defined higher dimensional Thompson groups as generalizations of the Thompson group $V = 1V$. By definition, n -dimensional Thompson group n_1V embeds into n_2V when $n_1 \leq n_2$. Brin [1] showed that V and $2V$ are not isomorphic. Bleak and Lanoue [3] showed n_1V and n_2V are isomorphic if and only if $n_1 = n_2$.

In [4], Bleak and Salazar-Díaz proved that $\mathbb{Z}^2 * \mathbb{Z}$ does not embed in V . Recently, Corwin and Haymaker [6] determined which right-angled Artin groups embed into V . Using the nonembedding result of [4], they showed that $\mathbb{Z}^2 * \mathbb{Z}$ is the only obstruction for a right-angled Artin group to be embedded into V . On the other hand, Belk, Bleak and Matucci [2] proved that a right-angled Artin group embeds in nV with sufficiently large n . They took n to be the sum of the number of vertices and the number of complementary edges in the defining graph. They conjectured that a right-angled Artin group embeds into $(n - 1)V$ if and only if the right-angled Artin group does not contain $\mathbb{Z}^n * \mathbb{Z}$. Corwin [5] constructed embeddings of $\mathbb{Z}^n * \mathbb{Z}$ into nV for every $n \geq 2$. It follows that every nV with $n \geq 2$ does not embed into V .

In this paper, we give another construction of embeddings of right-angled Artin groups into higher-dimensional Thompson groups. In particular, we may embed a right-angled Artin group into nV , where n is the number of

complementary edges in the defining graph. We may construct embeddings of $\mathbb{Z}^n * \mathbb{Z}$ into nV in this way.

The author would like to thank Takuya Sakasai and Tomohiko Ishida for helpful comments. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

2 Right-angled Artin groups

Let Γ be a finite graph with a vertex set $V(\Gamma) = \{v_i\}_{1 \leq i \leq m}$ and an edge set $E(\Gamma)$. The corresponding right-angled Artin group, denoted by A_Γ , is a group defined by the presentation

$$A_\Gamma = \langle g_1, \dots, g_m \mid g_i g_j = g_j g_i \text{ for all } \{v_i, v_j\} \in E(\Gamma) \rangle.$$

In the following, we let

$$\bar{E}(\Gamma) = \{\{v_i, v_j\} \mid v_i \neq v_j \in V(\Gamma) \text{ are not connected by edges.}\}$$

We call the elements of $\bar{E}(\Gamma)$ complementary edges.

We use the following theorem, known as the ping-pong lemma for the right-angled Artin groups.

Theorem 2.1 ([7]). *Let A_Γ be a right-angled Artin group with generators $\{g_i\}_{1 \leq i \leq m}$ acting on a set X . Suppose that there exist subsets S_i ($1 \leq i \leq m$) of X , with divisions $S_i = S_i^+ \amalg S_i^-$, satisfying the following conditions:*

- (1) $g_i(S_i^+) \subset S_i^+$ and $g_i^{-1}(S_i^-) \subset S_i^-$ for all i .
- (2) If g_i and g_j commute, then $g_i(S_j) = S_j$.
- (3) If g_i and g_j do not commute, then $g_i(S_j) \subset S_i^+$ and $g_i^{-1}(S_j) \subset S_i^-$.
- (4) There exists $x \in X - \bigcup_{i=1}^m S_i$ such that $g_i(x) \in S_i^+$ and $g_i^{-1}(x) \in S_i^-$ for all i .

Then this action is faithful.

3 Embedding right-angled Artin groups into nV

We use the following notations in [1]. We let I be a half-open interval $[0, 1)$. An n -dimensional rectangle is an affine copy of I^n in I^n , constructed by

repeating “dyadic divisions”. An n -dimensional pattern is a finite set of n -dimensional rectangles, with pairwise disjoint, non-empty interiors and whose union is I^n . A *numbered pattern* is a pattern with a one-to-one correspondence to $\{0, 1, \dots, r-1\}$ where r is the number of rectangles in the pattern.

Let $P = \{P_i\}_{0 \leq i \leq r-1}$ and $Q = \{Q_i\}_{0 \leq i \leq r-1}$ be numbered patterns. We define $v(P, Q)$ to be a map from I^n to itself which takes each P_i onto Q_i affinely so as to preserve the orientation. The n -dimensional Thompson group nV is the set of partially affine, partially orientation preserving right-continuous bijections from I^n to itself.

Using these notations, we give a construction of embeddings of right-angled Artin groups into higher dimensional Thompson groups.

Theorem 3.1. *Let Γ be a graph with the vertex set $V(\Gamma) = \{v_i\}_{1 \leq i \leq m}$. Suppose that there are nonempty subsets $\{D_i\}_{1 \leq i \leq m}$ of $\{1, \dots, n\}$, such that $D_i \cap D_j = \emptyset$ if and only if v_i and v_j are connected by an edge. Then the right-angled Artin group A_Γ embeds into nV .*

For a nonempty subset D of $\{1, \dots, n\}$, a D -slice of I^n is an n -dimensional rectangle $S = \prod_{d=1}^n I_d$, where $d \in D$ if and only if I_d is properly contained in $[0, 1)$.

Lemma 3.2. *Let D be a nonempty subset of $\{1, \dots, n\}$. For every D -slice S of I^n and every division $S = S^+ \amalg S^-$ where S^+ and S^- are again D -slices, there is $h \in nV$ satisfying*

- (1) h changes d -th coordinate of I^n if and only if $d \in D$.
- (2) $h(I^n - S^-) = S^+$ and $h^{-1}(I^n - S^+) = S^-$.

Proof. There is an n -dimensional pattern which contains S as a rectangle and consists of D -slices. We fix one of such pattern P , and consider $I^n - S$ as a disjoint union of $(|P| - 1)$ -many D -slices.

We divide S^+ into mutually disjoint $|P|$ -many D -slices. We choose one of those D -slices in S^+ and name it S^{++} . We consider $S^+ - S^{++}$ as a disjoint union of $(|P| - 1)$ -many D -slices. Similarly, we choose a D -slices S^{--} in S^- , and consider $S^- - S^{--}$ as a disjoint union of $(|P| - 1)$ -many D -slices.

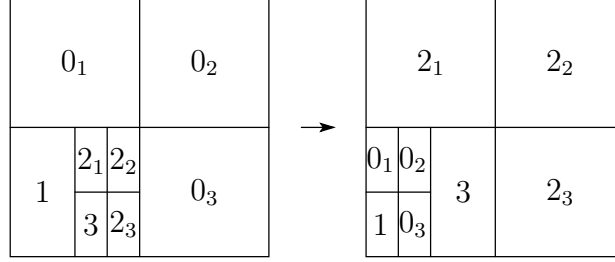
We define $h \in nV$ as follows:

0. h maps $I^n - S$ to $S^+ - S^{++}$.
1. h maps S^+ to S^{++} .
2. h maps $S^- - S^{--}$ to $I^n - S$.

3. h maps S^{--} to S^- .

This h satisfies conditions (1) and (2). \square

We show the construction of the map h in the following figure, in the case where $n = 2$, $D = \{1, 2\}$, $S = [0, 1/2) \times [0, 1/2)$ and $S^+ = [0, 1/4) \times [0, 1/2)$.



Remark 3.1. We take $h \in nV$ as in Lemma 3.2 with respect to a D -slice S and some division $S = S^+ \coprod S^-$. Let S' be a D' -slice with $D \cap D' = \emptyset$. We may observe that $h(S') = S'$, because S' is determined only by d' -th coordinates for $d' \in D'$, which are unchanged by h .

Lemma 3.3. For nonempty subsets $\{D_i\}_{1 \leq i \leq m}$ of $\{1, \dots, n\}$, there is a set of n -dimensional rectangles $\{S_i\}_{1 \leq i \leq m}$ satisfying

- (1) For every i , S_i is a D_i -slice of I^n .
- (2) $S_i \cap S_j = \emptyset$ if and only if $D_i \cap D_j \neq \emptyset$.
- (3) $\bigcup_{i=1}^m S_i \subsetneq I^n$.

Proof. We fix a dyadic division $I = \coprod_k J_k$, where $k \geq m + 1$. We define $S_i = \prod_{d=1}^n I_d^i$ by setting $I_d^i = J_i$ when $d \in D_i$, and $I_d^i = I$ otherwise.

- (1) Such S_i is a D_i -slice.
 - (2) If $D_i \cap D_j \neq \emptyset$, then $S_i \cap S_j = \emptyset$ since $I_d^i \cap I_d^j = J_i \cap J_j = \emptyset$ for all $d \in D_i \cap D_j$. The converse follows from the observation that a D -slice and a D' -slice always intersect when $D \cap D' = \emptyset$.
 - (3) Since we took J_k small enough, $\bigcup_{i=1}^m S_i$ is properly contained in I^n .
- Therefore, $\{S_i\}_{1 \leq i \leq m}$ satisfies conditions required in Lemma 3.3. \square

Proof of Theorem 3.1. Let Γ be a finite graph with vertices $\{v_i\}_{1 \leq i \leq m}$. Let $\{D_i\}_{1 \leq i \leq m}$ be nonempty subsets of $\{1, \dots, n\}$ such that $D_i \cap D_j = \emptyset$ if and only if v_i and v_j are connected by an edge.

According to Lemma 3.3, we take $\{S_i\}_{1 \leq i \leq m} \subset I^n$ with respect to $\{D_i\}_{1 \leq i \leq m}$. For every i , we fix D_i -slices S_i^+ and S_i^- satisfying $S_i = S_i^+ \coprod S_i^-$. We define h_i to be h of Lemma 3.2, which is defined with respect to S_i , S_i^+ and S_i^- .

We may define a homomorphism $\phi : A_\Gamma \rightarrow nV$ which maps each generator g_i , corresponding to the vertex v_i , to h_i . This homeomorphism is well-defined, since h_i and h_j commute when v_i and v_j are connected by an edge, according to the first condition of Lemma 3.2.

We consider an action of A_Γ on I^n , which is defined by $g \cdot x = \phi(g)(x)$. In the following, we show that this action is faithful, and thus ϕ is injective.

- (1) By the definition of h_i , $h_i(S_i^+) \subset S_i^+$ and $h_i^{-1}(S_i^-) \subset S_i^-$ for all i .
- (2) According to Remark 3.1, $h_i(S_j) = S_j$ when g_i and g_j commute.
- (3) When g_i and g_j do not commute, v_i and v_j are not connected by an edge, and S_i and S_j are disjoint. Therefore $h_i(S_j) \subset h_i(I^n - S_i) \subset S_i^+$ and $h_i^{-1}(S_j) \subset h_i^{-1}(I^n - S_i) \subset S_i^-$.
- (4) Since $\bigcup_{i=1}^m S_i \subsetneq I^n$, there is $x_0 \in I^n - S_i$ for all i . Such x_0 satisfies $h_i(x_0) \in S_i^+$ and $h_i^{-1}(x_0) \in S_i^-$, for all i .

By Theorem 2.1, ϕ is injective and an embedding of A_Γ into nV . \square

Corollary 3.4. *A right-angled Artin group A_Γ embeds into n -dimensional Thompson groups, where n is the number of complementary edges in Γ .*

Proof. We may assume that every vertex of Γ contributes to a complementary edge. In fact, if we let

$$V_0(\Gamma) = \{v \in V(\Gamma) \mid \text{For all } v \neq v' \in V(\Gamma), \{v, v'\} \in E(\Gamma)\},$$

then $A_\Gamma = \mathbb{Z}^{|V_0(\Gamma)|} \times A_{\Gamma'}$ for some subgraph Γ' satisfying the assumption and $\bar{E}(\Gamma) = \bar{E}(\Gamma')$. In general, if two groups G and H embed in nV , then $G \times H$ again embeds in nV . Therefore, it is enough to consider whether $A_{\Gamma'}$ embeds into nV or not.

Given A_Γ satisfying our assumption, we let $V(\Gamma) = \{v_i\}_{1 \leq i \leq m}$ be the vertex set and $\bar{E}(\Gamma) = \{\bar{e}_k\}_{1 \leq k \leq n}$ be the set of complementary edges. For every $i \in \{1, \dots, m\}$, we let

$$D_i = \{k \in \{1, \dots, n\} \mid \bar{e}_k \text{ contains } v_i \text{ as an endpoint.}\}.$$

We associate D_i with v_i .

Γ satisfies the condition required in Theorem 3.1, with respect to the subsets $\{D_i\}_{1 \leq i \leq m}$ of $\{1, \dots, n\}$. \square

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